

Vectors and Three-Dimensional Analytic Geometry

Chapters 1–11 dealt with single-variable calculus — differentiation and integration of functions $f(x)$ of one variable. In Chapters 12–15 we study multivariable calculus. Discussions of three-dimensional analytic geometry and vectors in Sections 12.1–12.5 prepare the way. In Sections 12.9–12.13, we differentiate and integrate vector functions, and apply the results to the geometry of curves in space and the motion of objects.

12.1 Rectangular Coordinates in Space

The coordinate of a point on the real line is its directed distance from the origin. Cartesian coordinates of a point in a plane are its directed distances from the coordinate axes. In space, Cartesian coordinates are directed distances from three fixed planes called the **coordinate planes**. In particular, we draw through a point O , called the **origin**, three mutually perpendicular lines called the x -, y -, and z -axes (Figure 12.1). Each of the axes is coordinatized with some unit distance (which need not be the same for all three axes). These three coordinate axes determine the three coordinate planes: The xy -coordinate plane is that plane containing the x - and y -axes, the yz -coordinate plane contains the y - and z -axes, and the xz -coordinate plane contains the x - and z -axes.

If P is any point in space, we draw lines from P perpendicular to the three coordinate planes (Figure 12.2). The directed distance from the yz -coordinate plane to P is parallel to the x -axis, and is called the x -coordinate of P . Similarly, y - and z -coordinates are defined as directed distances from the xz - and xy -coordinate planes to P . These three coordinates of P , written (x, y, z) , are called the **Cartesian** or **rectangular coordinates** of P . Note that if we draw lines through P that are perpendicular to the axes, then the directed distances from O to points of intersection of these perpendiculars with the axes are also the Cartesian coordinates of P (Figure 12.3).

FIGURE 12.1

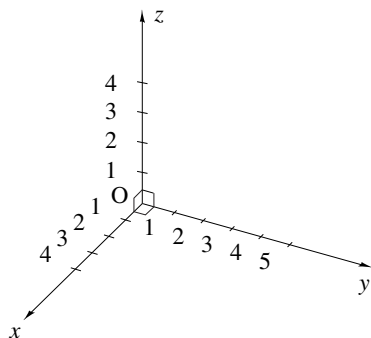


FIGURE 12.2

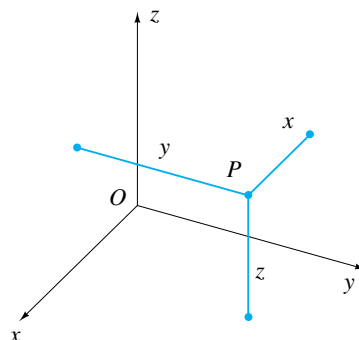
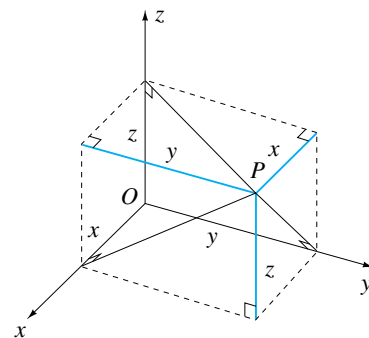
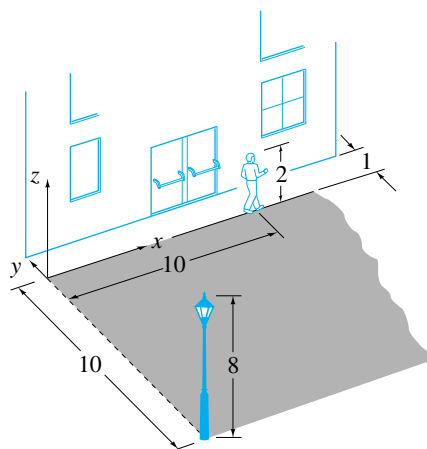


FIGURE 12.3



- ** 20. A man 2 m tall walks along the edge of a straight road 10 m wide (figure right). On the other edge of the road stands a streetlight 8 m high. A building runs parallel to the road and 1 m from it. If Cartesian coordinates are set up as shown (with x - and y -axes in the plane of the road), find coordinates of the tip of the man's shadow when he is at the position shown.



12.2 Curves and Surfaces

FIGURE 12.8

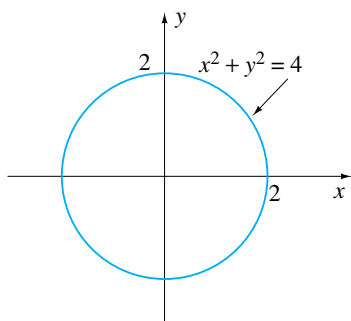
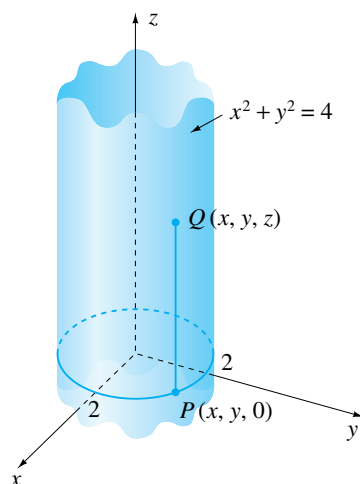


FIGURE 12.9



An equation involving the x - and y -coordinates of points in the xy -plane usually specifies a curve. For example, the equation $x^2 + y^2 = 4$ describes a circle of radius 2 centred at the origin (Figure 12.8). We now ask what is defined by an equation involving the Cartesian coordinates (x, y, z) of points in space. For example, the equation $z = 0$ describes all points in the xy -plane since all such points have a z -coordinate equal to zero. Similarly, $y = 2$ describes all points in the plane parallel to and 2 units to the right of the xz -plane. What does the equation $x^2 + y^2 = 4$ describe? In other words, regarded as a restriction on the x -, y -, and z -coordinates of points in space, rather than a restriction on the x - and y -coordinates of points in the xy -plane, what does it represent? Because the equation says nothing about z , there is no restriction whatsoever on z . In other words, the z -coordinate can take on all possible values, but x - and y -coordinates must be restricted by $x^2 + y^2 = 4$. If we consider those points in the xy -plane ($z = 0$) that satisfy $x^2 + y^2 = 4$, we obtain the circle in Figure 12.8. In space, each of these points has coordinates $(x, y, 0)$, where x and y still satisfy $x^2 + y^2 = 4$ (Figure 12.9). If we now take any point Q that is either directly above or directly below a point $P(x, y, 0)$ on this circle, it has exactly the same x - and y -coordinates as P ; only its z -coordinate differs. Thus the x - and y -coordinates of Q also satisfy $x^2 + y^2 = 4$. Since we can do this for any point P on the circle, it follows that $x^2 + y^2 = 4$ describes the right circular cylinder of radius 2 and infinite extent in Figure 12.9.

By reasoning similar to that used above, we can show that the equation $2x + y = 2$ describes the plane in Figure 12.10 parallel to the z -axis and standing on the straight line $2x + y = 2$, $z = 0$ in the xy -plane.

Finally, consider the equation $x^2 + y^2 + z^2 = 9$. Since $\sqrt{x^2 + y^2 + z^2}$ is the distance from the origin to a point with coordinates (x, y, z) , this equation describes all points that are 3 units away from the origin. In other words, $x^2 + y^2 + z^2 = 9$ describes points on a sphere of radius 3 centred at the origin.

It appears that one equation in the coordinates (x, y, z) of points in space specifies a surface. The shape of the surface is determined by the form of the equation. If one equation in the coordinates (x, y, z) specifies a surface, it is easy to see what two simultaneous equations specify. For instance, suppose we ask for all points in space whose coordinates satisfy both of the equations

$$x^2 + y^2 = 4, \quad z = 1.$$

By itself, $x^2 + y^2 = 4$ describes the cylinder in Figure 12.9. The equation $z = 1$ describes all points in a plane parallel to the xy -plane and 1 unit above it. To ask for all points that satisfy $x^2 + y^2 = 4$ and $z = 1$ simultaneously is to ask for all points that lie on both surfaces.

FIGURE 12.10

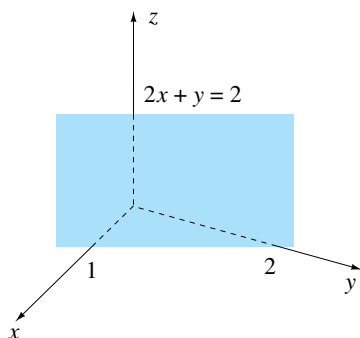
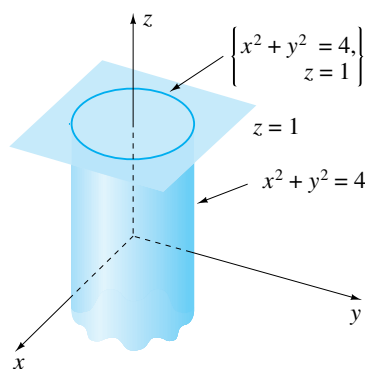


FIGURE 12.11



Consequently, the equations $x^2 + y^2 = 4$, $z = 1$ describe the curve of intersection of the two surfaces—the circle in Figure 12.11.

The equation $x = 0$ describes the yz -plane; the equation $y = 0$ describes the xz -plane. If we put the two equations together, $x = 0$ and $y = 0$, we obtain all points that lie on both the yz -plane and the xz -plane (i.e., the z -axis). In other words, equations for the z -axis are $x = 0$, $y = 0$.

Finally, $x^2 + y^2 + z^2 = 9$ is the equation of a sphere of radius 3 centred at the origin, and $y = 2$ is the equation of a plane parallel to the xz -plane and 2 units to the right. Together, the equations $x^2 + y^2 + z^2 = 9$, $y = 2$ describe the curve of intersection of the two surfaces—the circle in Figure 12.12. Note that by substituting $y = 2$ into the equation of the sphere, we can write alternatively that $x^2 + z^2 = 5$, $y = 2$. This pair of equations is equivalent to the original pair because all points that satisfy $x^2 + y^2 + z^2 = 9$, $y = 2$ also satisfy $x^2 + z^2 = 5$, $y = 2$, and vice versa. This new pair of equations provides an alternative way of visualizing the curve. Again $y = 2$ is the plane of Figure 12.12, but $x^2 + z^2 = 5$ describes a right circular cylinder of radius $\sqrt{5}$ and infinite extent around the y -axis (Figure 12.13). Our discussion has shown that the cylinder and plane intersect in the same curve as the sphere and plane.

In summary, we have illustrated that one equation in the coordinates (x, y, z) of a point specifies a surface; two simultaneous equations specify a curve, the curve of intersection of the two surfaces (provided, of course, that the surfaces do intersect).

In Chapters 2–10 we learned to appreciate the value of plotting and drawing curves in the xy -plane. Sometimes a plot or drawing serves as a device by which we can interpret algebraic

FIGURE 12.12

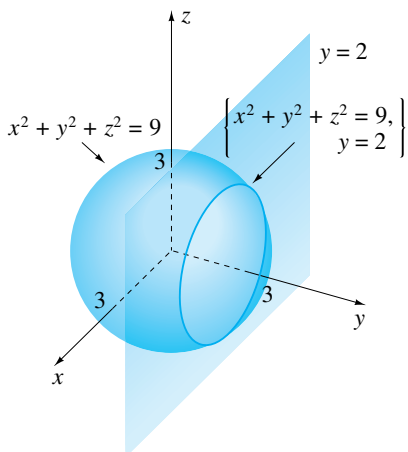


FIGURE 12.13

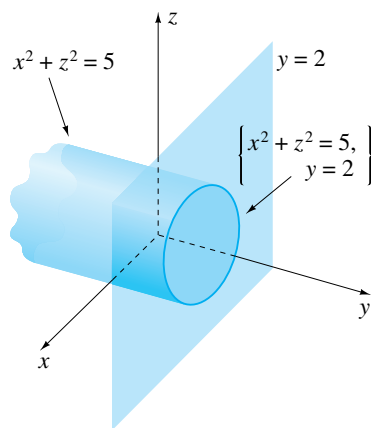


FIGURE 12.15a

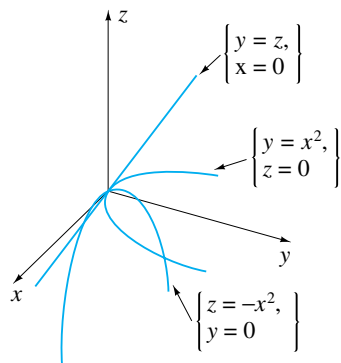


FIGURE 12.15b

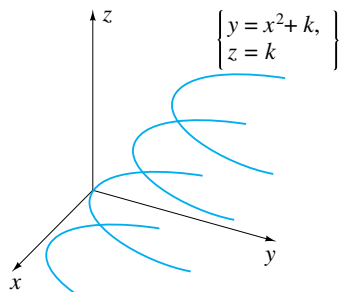
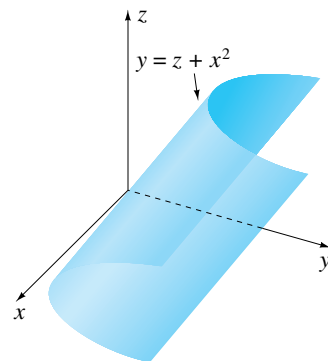


FIGURE 12.15c



We can sometimes “build” surfaces in much the same way that we “built” curves in single-variable calculus. For the surface $z = 1 - x^2 - y^2$, we first draw the surface $z = x^2 + y^2$ in Figure 12.14b. To draw $z = -(x^2 + y^2)$, we turn $z = x^2 + y^2$ upside down (Figure 12.16a), and finally we see that $z = 1 - x^2 - y^2$ is $z = -(x^2 + y^2)$ shifted upward 1 unit (Figure 12.16b).

FIGURE 12.16a

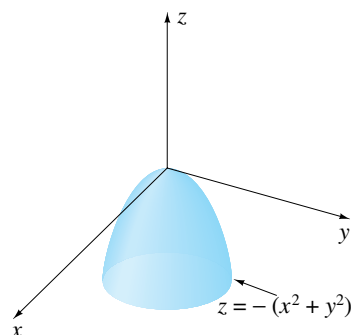
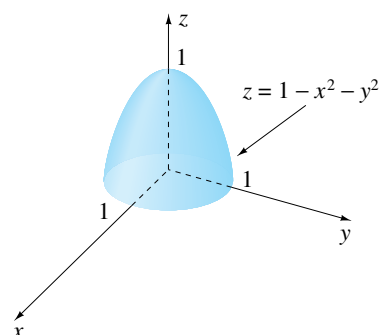


FIGURE 12.16b

**EXAMPLE 12.1**

Draw the surface defined by each of the following equations:

$$(a) z = \sqrt{4x + 2y - x^2 - y^2 - 4} \quad (b) y = 1 + \sqrt{x^2 + z^2}$$

SOLUTION

- (a) If we square the equation, and at the same time complete squares on $-x^2 + 4x$ and $-y^2 + 2y$, we have

$$z^2 = -(x - 2)^2 - (y - 1)^2 + 1,$$

or,

$$(x - 2)^2 + (y - 1)^2 + z^2 = 1.$$

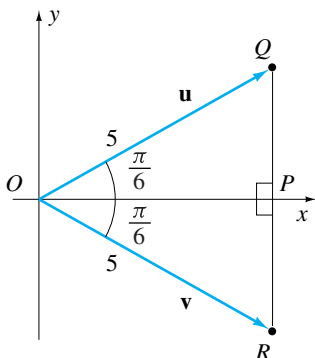
Because $\sqrt{(x - 2)^2 + (y - 1)^2 + z^2}$ is the distance from a point (x, y, z) to $(2, 1, 0)$, this equation states that (x, y, z) must always be a unit distance from $(2, 1, 0)$ [i.e., the equation $(x - 2)^2 + (y - 1)^2 + z^2 = 1$ defines a sphere of radius 1 centred at $(2, 1, 0)$] (Figure 12.17a). Because the original equation requires z to be nonnegative, the required surface is the upper half of this sphere — the hemisphere in Figure 12.17b.

THEOREM 12.1

Two vectors are equal if and only if they have the same components.

EXAMPLE 12.2

FIGURE 12.39



Find the components of a vector in the xy -plane that has length 5, its tail at the origin, and makes an angle of $\pi/6$ radians with the positive x -axis.

SOLUTION Figure 12.39 illustrates that there are two such vectors, \mathbf{u} and \mathbf{v} . From the triangles shown, it is clear that

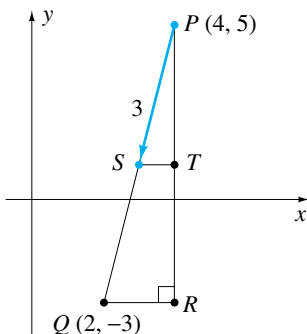
$$\|OP\| = 5 \cos(\pi/6) = 5\sqrt{3}/2 \quad \text{and} \quad \|PQ\| = \|PR\| = 5 \sin(\pi/6) = 5/2.$$

Consequently, Q and R have coordinates $Q = (5\sqrt{3}/2, 5/2)$ and $R = (5\sqrt{3}/2, -5/2)$, and

$$\mathbf{u} = \left(\frac{5\sqrt{3}}{2}, \frac{5}{2} \right), \quad \mathbf{v} = \left(\frac{5\sqrt{3}}{2}, -\frac{5}{2} \right).$$

EXAMPLE 12.3

FIGURE 12.40



Find the components of the vector in the xy -plane that has its tail at the point $(4, 5)$, has length 3, and points directly toward the point $(2, -3)$.

SOLUTION In Figure 12.40, $\|PQ\| = \sqrt{2^2 + 8^2} = 2\sqrt{17}$. Because of similar triangles, we can write that

$$\frac{\|ST\|}{\|PS\|} = \frac{\|QR\|}{\|PQ\|} \quad \text{or} \quad \|ST\| = \frac{3(2)}{2\sqrt{17}} = \frac{3}{\sqrt{17}}.$$

Similarly,

$$\|PT\| = \|PS\| \frac{\|PR\|}{\|PQ\|} = \frac{3(8)}{2\sqrt{17}} = \frac{12}{\sqrt{17}}.$$

Since $\|ST\|$ and $\|PT\|$ represent differences in the x - and y -coordinates of P and S (except for signs), the components of \mathbf{PS} are $(-3/\sqrt{17}, -12/\sqrt{17})$.

Unit Vectors and Scalar Multiplication

If the x -, y -, and z -components of a vector \mathbf{v} are (v_x, v_y, v_z) , often called the **Cartesian components** of \mathbf{v} , then these components represent the differences in the coordinates of its tip and tail. But then, according to equation 12.3, the length of the vector, which we denote by $|\mathbf{v}|$, is

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}. \quad (12.5)$$

In words, the length of a vector is the square root of the sum of the squares of its components.

DEFINITION 12.4

A vector \mathbf{v} is said to be a **unit vector** if it has length equal to 1 unit; that is, \mathbf{v} is a unit vector if

$$v_x^2 + v_y^2 + v_z^2 = 1. \quad (12.6)$$

To indicate that a vector is a unit vector, we place a circumflex $\hat{}$ above it: $\hat{\mathbf{v}}$.

12.8 Hanging Cables

Hanging cables are used in many engineering applications — suspension bridges, transmission lines, aerial gondolas, and so on. They are flexible members capable of withstanding tension forces but not shear forces. The shape of a hanging cable is determined by the loads that it supports, be they concentrated loads, distributed loads, or a combination of these.

Suppose the load per unit length in the x -direction (including the weight of the cable itself) of the cable in Figure 12.86 is $w(x)$ N/m. It is important to realize that this is not the load per unit length as measured along the cable. For simplicity, we have chosen a coordinate system where the lowest point of the cable is on the y -axis. Three forces act on that part of the cable between $x = 0$ and an arbitrary point x , a horizontal tension T_0 at $x = 0$, a tangential tension T at x , and the vertical load. They are in equilibrium so that we may equate horizontal and vertical components of their resultant to zero,

$$0 = -T_0 + T \cos \theta, \quad 0 = T \sin \theta - \int_0^x w(t) dt. \quad (12.47)$$

If we eliminate T , we obtain

$$T_0 \tan \theta = \int_0^x w(t) dt.$$

Since $\tan \theta$ is the slope of the cable at x , we can write

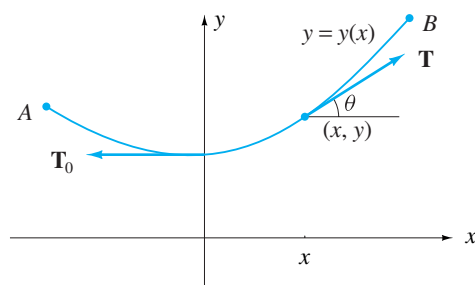
$$T_0 \frac{dy}{dx} = \int_0^x w(t) dt.$$

Since this equation is valid at every point on the cable we may differentiate with respect to x ,

$$\frac{d^2y}{dx^2} = \frac{w(x)}{T_0}. \quad (12.48)$$

Two integrations of this differential equation give the shape of the cable once the load $w(x)$ is determined.

FIGURE 12.86



The simplest load is $w(x) = w$, a constant. Uniformly distributed loads along the horizontal would be realized in a suspension bridge where the weight of the cable would be negligible compared to that of the roadway. Two integrations of 12.48 give

$$y(x) = \frac{wx^2}{2T_0} + C_1x + C_2.$$

Thus, uniformly loaded cables are parabolic. The fact that $y'(0) = 0$ implies that $C_1 = 0$, and therefore $y(x) = wx^2/(2T_0) + C_2$. The position of the x -axis determines C_2 . If the origin is chosen as the lowest point of the cable, then $C_2 = 0$ and $y(x) = wx^2/(2T_0)$.

THEOREM 12.8

If $\mathbf{v}(t) = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}}$, then

$$\frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}, \quad (12.57)$$

provided that the derivatives on the right exist.

PROOF If we substitute the components of $\mathbf{v}(t+h)$ and $\mathbf{v}(t)$ into Definition 12.14, then we have

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \lim_{h \rightarrow 0} \left\{ \frac{[v_x(t+h)\hat{\mathbf{i}} + v_y(t+h)\hat{\mathbf{j}} + v_z(t+h)\hat{\mathbf{k}}] - [v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}}]}{h} \right\} \\ &= \lim_{h \rightarrow 0} \left\{ \left[\frac{v_x(t+h) - v_x(t)}{h} \right] \hat{\mathbf{i}} + \left[\frac{v_y(t+h) - v_y(t)}{h} \right] \hat{\mathbf{j}} \right. \\ &\quad \left. + \left[\frac{v_z(t+h) - v_z(t)}{h} \right] \hat{\mathbf{k}} \right\} \end{aligned}$$

(according to equation 12.14), and

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \left[\lim_{h \rightarrow 0} \frac{v_x(t+h) - v_x(t)}{h} \right] \hat{\mathbf{i}} + \left[\lim_{h \rightarrow 0} \frac{v_y(t+h) - v_y(t)}{h} \right] \hat{\mathbf{j}} \\ &\quad + \left[\lim_{h \rightarrow 0} \frac{v_z(t+h) - v_z(t)}{h} \right] \hat{\mathbf{k}} \end{aligned}$$

(according to equation 12.54). Since each of the limits on the right exists, we can write that

$$\frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}.$$

Theorem 12.8 gives us a working rule for differentiating vector functions: To differentiate a vector function, we differentiate its Cartesian components. ■

EXAMPLE 12.37

If $\mathbf{v}(t) = t^2\hat{\mathbf{i}} + (3t^3 - 2t)\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, find $\mathbf{v}'(3)$.

SOLUTION According to 12.57,

$$\frac{d\mathbf{v}}{dt} = 2t\hat{\mathbf{i}} + (9t^2 - 2)\hat{\mathbf{j}}.$$

Consequently, $\mathbf{v}'(3) = 6\hat{\mathbf{i}} + 79\hat{\mathbf{j}}$.

The sum rule 3.9 for differentiation of scalar functions has its counterpart in the sum rule for vector functions,

$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt} \quad (12.58)$$

FIGURE 12.94

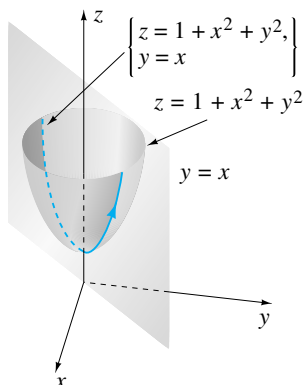
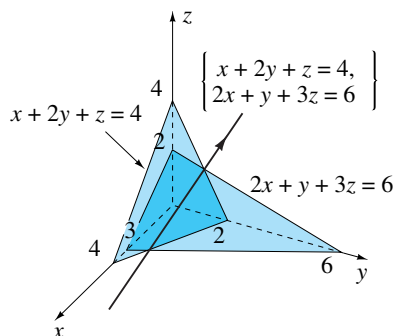


FIGURE 12.95



(b) The straight-line intersection of the two planes is shown in Figure 12.95. If we choose y as the parameter by setting $y = t$ (thus forcing y to increase as t increases), then

$$x + z = 4 - 2t, \quad 2x + 3z = 6 - t.$$

The solution of these equations for x and z in terms of t gives the parametric equations

$$x = 6 - 5t, \quad y = t, \quad z = -2 + 3t.$$

(c) The curve of intersection of the right circular cylinder $x^2 + (y - 1)^2 = 4$ and the plane $z = x$ is shown in Figure 12.96. We saw in Section 10.1 that trigonometric functions are particularly useful for circles. If we set $x = 2 \cos t$, then $y = 1 \pm 2 \sin t$. A set of parametric equations for the curve is therefore

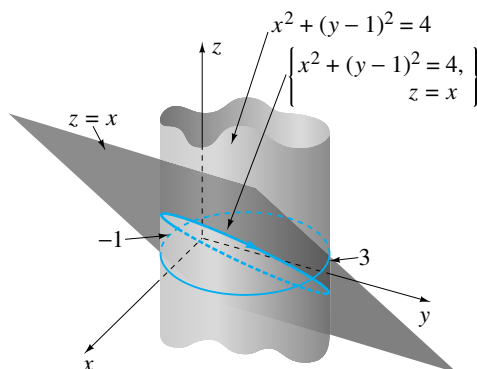
$$x = 2 \cos t, \quad y = 1 + 2 \sin t, \quad z = 2 \cos t, \quad 0 \leq t \leq 2\pi.$$

Any range of values of t of length 2π traces the curve exactly once. To check that these equations specify the correct direction along the curve, we note that $t = 0$ gives the point $(2, 1, 2)$ and $t = \pi/2$ gives $(0, 3, 0)$. Since values of t between 0 and $\pi/2$ give one-quarter of the curve (rather than three-quarters), points are indeed generated in the required direction indicated by the arrowhead in Figure 12.96. Had we chosen the equations

$$x = 2 \cos t, \quad y = 1 - 2 \sin t, \quad z = 2 \cos t, \quad 0 \leq t \leq 2\pi,$$

we would have generated the same set of points traced in the opposite direction.

FIGURE 12.96



Since $\hat{\mathbf{T}} = d\mathbf{r}/ds$ and $\hat{\mathbf{N}} = \kappa^{-1}d\hat{\mathbf{T}}/ds = \kappa^{-1}d^2\mathbf{r}/ds^2$, it follows that

$$\begin{aligned}\tau &= -\frac{1}{\kappa} \frac{d^2\mathbf{r}}{ds^2} \cdot \frac{d\mathbf{r}}{ds} \times \left(\frac{1}{\kappa} \frac{d^3\mathbf{r}}{ds^3} - \frac{1}{\kappa^2} \frac{d\kappa}{ds} \frac{d^2\mathbf{r}}{ds^2} \right) \\ &= -\frac{1}{\kappa^2} \frac{d^2\mathbf{r}}{ds^2} \cdot \frac{d\mathbf{r}}{ds} \times \frac{d^3\mathbf{r}}{ds^3} + \frac{1}{\kappa^3} \frac{d\kappa}{ds} \frac{d^2\mathbf{r}}{ds^2} \cdot \frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2} \\ &= \frac{1}{\kappa^2} \frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3},\end{aligned}\tag{12.96}$$

which expresses τ in terms of s . To express it in terms of an arbitrary parameter t along the curve rather than length s , we use

$$\begin{aligned}\frac{d\mathbf{r}}{ds} &= \frac{dt}{ds} \frac{d\mathbf{r}}{dt} = \frac{dt}{ds} \dot{\mathbf{r}}, \\ \frac{d^2\mathbf{r}}{ds^2} &= \frac{d^2t}{ds^2} \frac{d\mathbf{r}}{dt} + \frac{dt}{ds} \frac{d}{dt} \left(\frac{d\mathbf{r}}{dt} \right) \frac{dt}{ds} = \frac{d^2t}{ds^2} \dot{\mathbf{r}} + \left(\frac{dt}{ds} \right)^2 \ddot{\mathbf{r}}, \\ \frac{d^3\mathbf{r}}{ds^3} &= \frac{d^3t}{ds^3} \frac{d\mathbf{r}}{dt} + \frac{d^2t}{ds^2} \frac{d}{dt} \left(\frac{d\mathbf{r}}{dt} \right) \frac{dt}{ds} + 2 \left(\frac{dt}{ds} \right) \left(\frac{d^2t}{ds^2} \right) \frac{d^2\mathbf{r}}{dt^2} + \left(\frac{dt}{ds} \right)^2 \frac{d}{dt} \left(\frac{d^2\mathbf{r}}{dt^2} \right) \frac{dt}{ds} \\ &= \frac{d^3t}{ds^3} \dot{\mathbf{r}} + 3 \frac{d^2t}{ds^2} \frac{dt}{ds} \ddot{\mathbf{r}} + \left(\frac{dt}{ds} \right)^3 \dddot{\mathbf{r}}.\end{aligned}$$

With these,

$$\begin{aligned}\tau &= \frac{1}{\kappa^2} \left(\frac{dt}{ds} \dot{\mathbf{r}} \right) \cdot \left[\frac{d^2t}{ds^2} \dot{\mathbf{r}} + \left(\frac{dt}{ds} \right)^2 \ddot{\mathbf{r}} \right] \times \left[\frac{d^3t}{ds^3} \dot{\mathbf{r}} + 3 \frac{d^2t}{ds^2} \frac{dt}{ds} \ddot{\mathbf{r}} + \left(\frac{dt}{ds} \right)^3 \dddot{\mathbf{r}} \right] \\ &= \frac{1}{\kappa^2} \left(\frac{dt}{ds} \right)^6 (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} \times \dddot{\mathbf{r}}) = \frac{|\dot{\mathbf{r}}|^6}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} \frac{1}{|\dot{\mathbf{r}}|^6} (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} \times \dddot{\mathbf{r}}),\end{aligned}$$

and hence,

$$\tau(t) = \frac{\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} \times \dddot{\mathbf{r}}}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2}.\tag{12.97}$$

The **third Frenet–Serret formula** expresses $d\hat{\mathbf{N}}/ds$ in terms of $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$; it is developed in Exercise 29.

EXERCISES 12.12

In Exercises 1–5 find $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ at each point on the curve.

- $x = \sin t$, $y = \cos t$, $z = t$, $-\infty < t < \infty$
- $x = t$, $y = t^2$, $z = t^3$, $t \geq 1$
- $x = (t-1)^2$, $y = (t+1)^2$, $z = -t$, $-3 \leq t \leq 4$
- $x + y = 5$, $x^2 - y = z$, from $(5, 0, 25)$ to $(0, 5, -5)$
- $z = x$, $x^2 + y^2 = 4$, $y \geq 0$, from $(2, 0, 2)$ to $(-2, 0, -2)$

In Exercises 6–10 find $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ at the point.

- $x = 4 \cos t$, $y = 6 \sin t$, $z = 2 \sin t$, $-\infty < t < \infty$; $(2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$
- $x = 2 - 5t$, $y = 1 + t$, $z = 6 + 4t^3$, $-\infty < t < \infty$; $(7, 0, 2)$
- $x^2 + y^2 + z^2 = 4$, $z = \sqrt{x^2 + y^2}$, directed so that x increases when y is positive; $(1, 1, \sqrt{2})$
- $x = y^2 + 1$, $z = x + 5$, directed so that y increases along the curve; $(5, 2, 10)$
- $x^2 + (y-1)^2 = 4$, $x = z$, directed so that z decreases when y is negative; $(2, 1, 2)$

Thus,

$$\frac{d\hat{\mathbf{T}}}{dt} = (\cos \theta, \sin \theta) \frac{d\theta}{dt} \implies \left| \frac{d\hat{\mathbf{T}}}{dt} \right| = -\frac{d\theta}{dt}.$$

Substitution into $-N + 9.81m \sin \theta = m|\mathbf{v}||d\hat{\mathbf{T}}/dt|$ gives

$$-N + 9.81m \sin \theta = m \left(-a \frac{d\theta}{dt} \right) \left(-\frac{d\theta}{dt} \right) = ma \left(\frac{d\theta}{dt} \right)^2.$$

Replacing $(d\theta/dt)^2$ by $(19.62/a)(1 - \sin \theta)$, we obtain

$$-N + 9.81m \sin \theta = 19.62m(1 - \sin \theta) \implies N = 9.81m(3 \sin \theta - 2).$$

Clearly, $N = 0$ when $\theta = \sin^{-1}(2/3)$, and this is the angle at which the mass leaves the sphere. Its speed at this point is

$$|\mathbf{v}| = -a \frac{d\theta}{dt} = a \sqrt{\frac{19.62}{a} \left(1 - \frac{2}{3} \right)} = \sqrt{\frac{19.62a}{3}}.$$

EXERCISES 12.13

In Exercises 1–5 find the velocity, speed, and acceleration of a particle if the given equations represent its position as a function of time.

- $x(t) = \sqrt{t^2 + 1}$, $y(t) = t\sqrt{t^2 + 1}$, $t \geq 0$
- $x(t) = t + 1/t$, $y(t) = t - 1/t$, $t \geq 1$
- $x(t) = \sin t$, $y(t) = 3 \cos t$, $z(t) = \sin t$, $0 \leq t \leq 10\pi$
- $x(t) = t^2 + 1$, $y(t) = 2te^t$, $z(t) = 1/t^2$, $1 \leq t \leq 5$
- $x(t) = e^{-t^2}$, $y(t) = t \ln t$, $z(t) = 5$, $t \geq 1$

In Exercises 6–7 a particle at $(1, 2, -1)$ starts from rest at time $t = 0$. Find its position as a function of time if the given function defines its acceleration.

- $\mathbf{a}(t) = 3t^2\hat{\mathbf{i}} + (t + 1)\hat{\mathbf{j}} - 4t^3\hat{\mathbf{k}}$, $t \geq 0$
- $\mathbf{a}(t) = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}/(t + 1)^3$, $t \geq 0$

In Exercises 8–9 find the tangential and normal components of acceleration for a particle moving with position defined by the given functions (where t is time).

- $x(t) = t$, $y(t) = t^2 + 1$, $t \geq 0$
- $x(t) = \cos t$, $y(t) = \sin t$, $z = t$, $t \geq 0$

10. Show that the normal component of acceleration of a particle can be expressed in the form $a_N = |\mathbf{v}|^2/\rho = \kappa|\mathbf{v}|^2$.

11. Find the kinetic energy for each particle in Exercises 1–5 if its mass is 2 g. Assume that x , y , and z are measured in metres and t in seconds.

12. A particle starts at the origin and moves along the curve $4y = x^2$ to the point $(4, 4)$.

- If the y -component of its acceleration is always equal to 2 and the y -component of its velocity is initially zero, find the x -component of its acceleration.
- If the x -component of its acceleration is equal to $24t^2$ (t being time) and the x -component of its velocity is initially zero, find the y -component of its acceleration.

13. A particle moves along the curve $x(t) = t$, $y(t) = t^3 - 3t^2 + 2t$, $0 \leq t \leq 5$ in the xy -plane (where t is time). Is there any point at which its velocity is parallel to its displacement?

14. A particle moves along the curve $y = x^3 - 2x + 3$ so that its x -component of velocity is always equal to 5. Find its acceleration.

15. If a particle starts at time $t = 0$ from rest at position $(3, 4)$ and experiences an acceleration $\mathbf{a} = -5t^4\hat{\mathbf{i}} - (2t^3 + 1)\hat{\mathbf{j}}$, find its speed at $t = 2$.

- * 16. A particle travels counterclockwise around the circle $(x - h)^2 + (y - k)^2 = R^2$ in the figure below. Show that the speed of the particle at any time is $|\mathbf{v}| = \omega R$, where $\omega = d\theta/dt$ is called the *angular speed* of the particle.

